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Quantile Coherency: A General Measure for Dependence between Cyclical Economic Variables

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Summary In this paper, we introduce quantile coherency to measure general dependence structures emerging in the joint distribution in the frequency domain and argue that this type of dependence is natural for economic time series but remains invisible when only the traditional analysis is employed. We define estimators which capture the general dependence structure, provide a detailed analysis of their asymptotic properties and discuss how to conduct inference for a general class of possibly nonlinear processes. In an empirical illustration we examine the dependence of bivariate stock market returns and shed new light on measurement of tail risk in financial markets. We also provide a modelling exercise to illustrate how applied researchers can benefit from using quantile coherency when assessing time series models.

Keywords: *Cross-spectral analysis, Ranks, Copula, Stock market, Risk.*

1. DEPENDENCE STRUCTURES BEYOND SECOND-ORDER MOMENTS

One of the fundamental problems faced by a researcher in economics is how to quantify the dependence between economic variables. Although correlated variables are rather commonly observed phenomena in economics, it is often the case that strongly correlated variables under study are truly independent, and what we measure is mere spurious correlation; see Granger and Newbold (1974). Conversely, but equally deluding, uncorrelated variables may possess dependence in different parts of the joint distribution, and/or at different frequencies. This dependence stays hidden when classical measures based on linear correlation and traditional cross-spectral analysis are used; see Croux et al. (2001), Ning and Chollete (2009) and Fan and Patton (2014). Hence, conventional models derived from averaged quantities as for example covariance-based measures may deliver rather misleading results.

In this paper, we introduce a new class of cross-spectral densities that characterise the dependence in quantiles of the joint distribution across frequencies (i. e., with respect to cycles). Subsequently, standardisation of the before-mentioned quantile spectra yields a related quantity to which we will refer to as quantile coherency. We define and motivate the quantile-based cross-spectral quantities in analogy to their traditional counterparts. Yet, instead of quantifying dependence in terms of joint moments (i. e., by averaging with respect to the joint distribution), the new measures are defined in terms of the probabilities to exceed quantiles. Hence, they are designed to detect any general type of dependence structure that may arise between variables under study.

Such complex dynamics may arise naturally in many macroeconomic, or financial time series such as growth rates, inflation, housing markets, or stock market returns. In financial markets, extremely scarce and negative events in one asset can cause irrational outcomes and panics leading investors to ignore economic fundamentals and cause similarly extreme negative outcomes in other assets. In such situations, markets may be connected more strongly than in calm periods of small, or positive returns; cf. Bae et al. (2003). Hence, the co-occurrences of large negative values may be more common across stock markets than co-occurrences of large positive values reflecting asymmetric behaviour of economic agents. Moreover, long-term fluctuations in quantiles of the joint distribution may differ from the ones in the short-term due to differing risk perception of economic agents over distinct investment horizons. This behaviour produces various degrees of persistence at different parts of the joint distribution, while on average the stock market returns remain impersistent. In univariate macroeconomic variables, researchers document asymmetric adjustment paths (cf. Neftci (1984) and Enders and Granger (1998)) as firms are more prone to an increase than to a decrease in prices. Asymmetric business cycle dynamics at different quantiles can be caused by positive shocks to output being more persistent than negative shocks. While output fluctuations are known to be persistent, Beaudry and Koop (1993) document less persistence at longer horizons. Such asymmetric dependence at different horizons can be shared by multiple variables. Because classical, covariance-based approaches only take averaged information into account, these types of dependence fail to be identified by traditional means. Revealing such dependence structures, quantile cross-spectral analysis introduced in this paper can fundamentally change the way how we view the dependence between economic time series, and opens new possibilities for the modelling of interactions between economic and financial variables.

Quantile cross-spectral analysis provides a general, unifying framework for estimating dependence between economic time series. As noted in the early work of Granger (1966), the spectral distribution of an economic variable has a typical shape which distinguishes long-term fluctuations from short-term ones. These fluctuations point to economic activity at different frequencies (after removal of trend in mean, as well as seasonal components). After Granger (1966) studied the behaviour of single time series, important literature using cross-spectral analysis to identify the dependence between variables quickly emerged (from Granger (1969) to more recent Croux et al. (2001)). Instead of considering only cross-sectional correlations, researchers started to use coherency (frequency dependent correlation) to investigate short-run and long-run dynamic properties of multiple time series, and identify business cycle synchronisation; see Croux et al. (2001). In one of his very last papers, Granger (2010) hypothesised about possible cointegrating relationships in quantiles, leading to the first notion of general types of dependence that quantile cross-spectral analysis is addressing. The quantile cointegration developed by Xiao (2009) partially addresses the problem, but does not allow to fully explore the frequency dependent structure of correlations in different quantiles of the joint distribution.

Three toy examples illustrating the potential offered by quantile cross-spectral analysis are depicted in Figure 1. In each example one distinct type of dependence is considered: cross-sectional dependence (left), serial dependence (centre), and independence (right). We consider bivariate processes (x_t, y_t) that possess the desired dependence structure, but are indistinguishable in terms of traditional coherency. In the examples, (ϵ_t) is an independent sequence of standard normally distributed random variables. In the left column of Figure 1 the dependence emerging between ϵ_t and ϵ_t^2 is depicted. It is important to observe that ϵ_t and ϵ_s^2 are uncorrelated. Therefore, traditional coherency for $(\epsilon_t, \epsilon_t^2)$

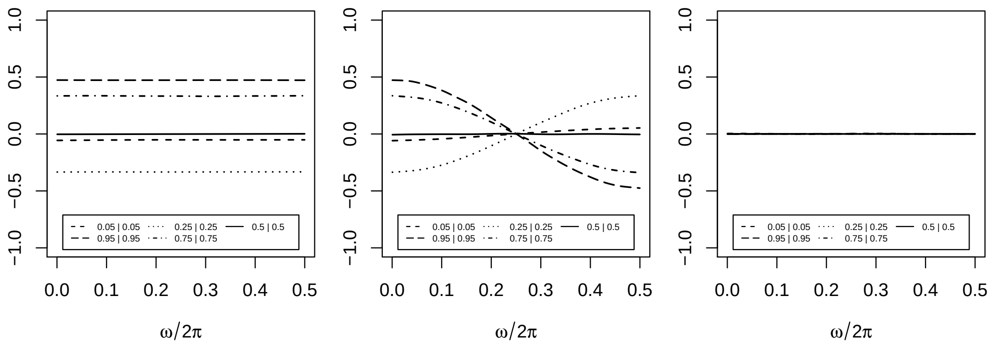

Figure 1. Illustration of dependence between processes x_t and y_t.

would read zero across all frequencies, even though it is obvious that ϵ_t and ϵ_t^2 are dependent. From the newly introduced quantile coherency, this dependence can easily be observed. More precisely, we can distinguish various degrees of dependence for each part of the distribution. For example, there is no dependence in the centre of the distribution (i. e., 0.5|0.5), but when the quantile levels are different from 0.5 the dependence becomes visible.[1] In this example the quantile coherency is constant across frequencies, which corresponds to the fact that there is no serial dependence. In the centre column of Figure 1 the process $(\epsilon_t, \epsilon_{t-1}^2)$ is studied, where we have introduced a time lag. Intuitively, the dependence in quantiles of this bivariate process will be the same as in the previous example (left column) in the long run, referring to frequencies close to zero. With increasing frequency, dependence will decline or incline gradually to values with opposite signs, as high frequency movements are in opposition due to the lag shift. This is clearly captured by quantile coherency, while the dependence structure would stay hidden away from traditional coherency, again, as it averages the dependence across quantiles. We can think about these processes as being "spuriously independent". To demonstrate the behaviour of the quantile coherency when the processes under consideration are truly independent, we observe in the right column of Figure 1 the quantities for independent bivariate Gaussian white noise, where quantile coherency displays zero dependence at all quantiles and frequencies, as expected. These illustrations strongly support our claim that there is need for more general measures that can provide a better understanding of the dependence between variables. These very simple, yet illuminating motivating examples focus on uncovering dependence in uncorrelated variables. Later in the text (Section 6), we further discuss a data generating process based on quantile vector autoregression (QVAR), which is able to generate even richer dependence structures, revealing once more the limitations of the traditional approach. In Figure 2, the real part of the quantile coherencies of the QVAR(1), QVAR(2) and QVAR(3) example processes are shown. Further, in Section S3, we discuss how to interpret quantile coherency in the special cases of bivariate Gaussian VAR(1).

This paper is organised as follows. In Section 2 we introduce notation, define quantile coherency and an estimator for it. In Section 3 we discuss the proposed methodology and related literature. In Section 4 we provide a rigorous asymptotic analysis of the

[1] All plots show real parts of the complex-valued quantities for illustratory purposes. Further discussion on how to interpret the real part and the imaginary part of quantile coherency are deferred to Section 3.

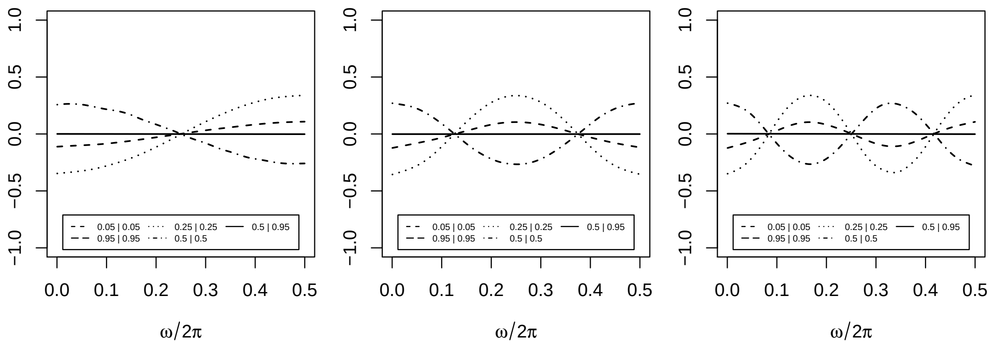

Figure 2. Illustration of dependence between vector quantile autoregressive processes.

estimator's statistical properties. In Section 5, to support our theoretical discussions empirically, we employ the new methodology to inspect bivariate stock market returns, one of the most prominent time series in economics, and reveal dependencies in cycles of quantile-based features. We continue our empirical study in Section 6 by using quantile coherency to compare time series models with respect to their capabilities to capture the revealed dependencies. In the supplementary material to this paper (available from the publisher's homepage), we discuss additional quantile-based cross-spectral quantities (Section S1), discuss quantile vector autoregressive processes as examples with rich dynamics (Section S2), discuss how the new, quantile-based spectral quantities and their traditional counterparts are related (Section S3), state additional theoretical results (Section S4), comment on the construction of the interval estimators (Section S5), and provide rigorous proofs for all theoretical results (Section S6).

2. QUANTILE CROSS-SPECTRAL QUANTITIES AND THEIR ESTIMATORS

Throughout the paper $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ denotes a d-variate, strictly stationary process, with components $X_{t,j}$, $j = 1, \dots, d$; i. e. $\boldsymbol{X}_t = (X_{t,1}, \dots, X_{t,d})'$. The marginal distribution function of $X_{t,j}$ will be denoted by F_j, and by $q_j(\tau) := F_j^{-1}(\tau) := \inf\{q \in \mathbb{R} : \tau \le F_j(q)\}$, where $\tau \in [0, 1]$, we denote the corresponding quantile function. We use the convention $\inf \emptyset = +\infty$, such that, if $\tau = 0$ or $\tau = 1$, then $-\infty$ and $+\infty$ are possible values for $q_j(\tau)$, respectively. We will write $\overline{z}$ for the complex conjugate, $\Re z$ for the real part and $\Im z$ for the imaginary part of $z \in \mathbb{C}$, respectively. The transpose of a matrix $\boldsymbol{A}$ will be denoted by $\boldsymbol{A}'$, the inverse of a regular matrix $\boldsymbol{B}$ will be denoted by $\boldsymbol{B}^{-1}$.

As a measure for the serial and cross-dependency structure of $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$, we define the matrix of quantile cross-covariance kernels, $\Gamma_k(\tau_1, \tau_2) := (\gamma_k^{j_1,j_2}(\tau_1, \tau_2))_{j_1,j_2=1,\dots,d}$, where

$$\gamma_k^{j_1,j_2}(\tau_1, \tau_2) := \mathrm{Cov}\Big(I\{X_{t+k,j_1} \le q_{j_1}(\tau_1)\}, I\{X_{t,j_2} \le q_{j_2}(\tau_2)\}\Big), \tag{2.1}$$

$j_1, j_2 \in \{1, \dots, d\}$, $k \in \mathbb{Z}$, $\tau_1, \tau_2 \in [0, 1]$, and $I\{A\}$ denotes the indicator function of the event A. In the frequency domain this yields (under appropriate mixing conditions) the matrix of quantile cross-spectral density kernels $\mathfrak{f}(\omega; \tau_1, \tau_2) := (\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2))_{j_1,j_2=1,\dots,d}$, where

$$\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2) := (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \gamma_k^{j_1,j_2}(\tau_1, \tau_2) \mathrm{e}^{-\mathrm{i}k\omega}, \tag{2.2}$$

$j_1, j_2 \in \{1, \ldots, d\}$, $\omega \in \mathbb{R}$, $\tau_1, \tau_2 \in [0,1]$. A closely related quantity that can be used as a measure for the dynamic dependence of the two processes $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$ is the quantile coherency kernel of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$, which we define as

$$\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2) := \frac{\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)}{\Big(\mathfrak{f}^{j_1,j_1}(\omega;\tau_1,\tau_1)\mathfrak{f}^{j_2,j_2}(\omega;\tau_2,\tau_2)\Big)^{1/2}}, \tag{2.3}$$

$(\tau_1, \tau_2) \in (0,1)^2$. We define the estimator for the quantile cross-spectral density as the collection

$$I^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2) := \frac{1}{2\pi n} d^{j_1}_{n,R}(\omega;\tau_1) d^{j_2}_{n,R}(-\omega;\tau_2), \tag{2.4}$$

$j_1, j_2 = 1, \ldots, d$, $\omega \in \mathbb{R}$, $(\tau_1,\tau_2) \in [0,1]^2$, and call it the rank-based copula cross-periodograms, shortly, the CCR-periodograms, where

$$d^{j}_{n,R}(\omega;\tau) := \sum_{t=0}^{n-1} I\{\hat{F}_{n,j}(X_{t,j}) \le \tau\} \mathrm{e}^{-\mathrm{i}\omega t} = \sum_{t=0}^{n-1} I\{R_{n;t,j} \le n\tau\} \mathrm{e}^{-\mathrm{i}\omega t},$$

$j = 1, \ldots, d$, $\omega \in \mathbb{R}$, $\tau \in [0,1]$, and $\hat{F}_{n,j}(x) := n^{-1}\sum_{t=0}^{n-1} I\{X_{t,j} \le x\}$ denotes the empirical distribution function of $X_{t,j}$ and $R_{n;t,j}$ denotes the (maximum) rank of $X_{t,j}$ among $X_{0,j}, \ldots, X_{n-1,j}$. We will denote the matrix of CCR-periodograms by

$$\boldsymbol{I}_{n,R}(\omega;\tau_1,\tau_2) := (I^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2))_{j_1,j_2=1,\ldots,d}. \tag{2.5}$$

From the univariate case it is already known (cf. Proposition 3.4 in Kley et al. (2016)) that the CCR-periodograms fail to estimate $\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ consistently. Consistency can be achieved by smoothing $I^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)$ across frequencies. More precisely, we consider

$$\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2) := \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) I^{j_1,j_2}_{n,R}(2\pi s/n, \tau_1, \tau_2), \tag{2.6}$$

where W_n denotes a sequence of weight functions, precisely to be defined in Section 4.

We will denote the matrix of smoothed CCR-periodograms by

$$\hat{\boldsymbol{G}}_{n,R}(\omega;\tau_1,\tau_2) := (\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2))_{j_1,j_2=1,\ldots,d}. \tag{2.7}$$

The estimators for the quantile coherency is then given by

$$\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2) := \frac{\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)}{\Big(\hat{G}^{j_1,j_1}_{n,R}(\omega;\tau_1,\tau_1)\hat{G}^{j_2,j_2}_{n,R}(\omega;\tau_2,\tau_2)\Big)^{1/2}}. \tag{2.8}$$

In Section 4 we will prove that

$$\hat{\boldsymbol{\mathfrak{R}}}_{n,R}(\omega;\tau_1,\tau_2) := \big(\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)\big)_{j_1,j_2=1,\ldots,d}$$

is a legitimate estimator for $\boldsymbol{\mathfrak{R}}(\omega;\tau_1,\tau_2) := \big(\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)\big)_{j_1,j_2=1,\ldots,d}$, the matrix of quantile coherencies.

3. DISCUSSION OF THE INTRODUCED QUANTITIES AND ESTIMATORS

The quantile-based quantities defined in Section 2 are functions of the two variables τ_1 and τ_2. They are thus richer in information than the traditional counterparts. We have added the term kernel to the name for the quantities to stress this fact, but will frequently omit it in the rest of the paper, for the sake of brevity. For continuous F_{j_1} and F_{j_2}, the quantile cross-covariances defined in (2.1) coincide with the difference of the copula of (X_{t+k,j_1}, X_{t,j_2}) and the independence copula. Thus, they provide important information about both the serial dependence (by letting k vary) and the cross-section-dependence (by choosing $j_1 \neq j_2$). For the quantile cross-spectral density we have

$$\int_{-\pi}^{\pi} \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)\mathrm{e}^{\mathrm{i}k\omega}\mathrm{d}\omega + \tau_1\tau_2 = \mathbb{P}\Big(X_{t+k,j_1} \leq q_{j_1}(\tau_1), X_{t,j_2} \leq q_{j_2}(\tau_2)\Big), \tag{3.9}$$

where the quantity on the right hand side, as a function of (τ_1, τ_2), is again the copula of the pair (X_{t+k,j_1}, X_{t,j_2}). The equality (3.9) thus shows how any of the pair copulas can be derived from the quantile cross-spectral density kernel defined in (2.2). Thus, the quantile cross-spectral density kernel provides a full description of all copulas of pairs in the process. Comparing these new quantities with their traditional counterparts, it can be observed that covariances and means are essentially replaced by copulas and quantiles. Similar to the regression setting, where this approach provides valuable extra information (cf. Koenker (2005)), the quantile-based approach to spectral analysis supplements the traditional L^2-spectral analysis.

Observe that $\mathfrak{R}$ takes values in $\mathbb{C}^{d\times d}$ (the set of all complex-valued $d \times d$ matrices). Further, note that, as a function of ω, but for fixed τ_1, τ_2, it coincides with the traditional coherency of the bivariate, binary process

$$\Big(I\{X_{t,j_1} \leq q_{j_1}(\tau_1)\}, I\{X_{t,j_2} \leq q_{j_2}(\tau_2)\}\Big)_{t\in\mathbb{Z}}. \tag{3.10}$$

The time series in (3.10) has the bivariate time series $(X_{t,j_1}, X_{t,j_2})_{t\in\mathbb{Z}}$ as a "latent driver" and indicates whether the values of the components j_1 and j_2 are below the respective marginal distribution's τ_1- and τ_2-quantile.

Note the important fact that $\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ is undefined when (τ_1,τ_2) is on the boundary of $[0,1]^2$. By Cauchy-Schwarz inequality, we further observe that the range of possible values is limited to $\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2) \in \{z \in \mathbb{C} : |z| \leq 1\}$. Note that, as (τ_1,τ_2) approaches the border of the unit square, the quantile cross-spectral density vanishes. Therefore, quantile coherency is better suited to measure dependence of extremes than the quantile cross-spectral density (which is not standardised). Implicitly, we take advantage of the fact that the quantile cross-spectral density and quantile spectral densities vanish at the same rate and therefore the quotient yields a meaningful quantity when the quantile levels (τ_1,τ_2) approaches the border of the unit square.

The quantile coherency kernel contains very valuable information about the joint dynamics of the time series $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$. In contrast to the traditional case, where coherency will always equal one if $j_1 = j_2 =: j$, the quantile-based versions of these quantities are capable of delivering valuable information about one single component of $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ as well. Quantile coherency then quantifies the joint dynamics of $(I\{X_{t,j} \leq q_j(\tau_1)\})_{t\in\mathbb{Z}}$ and $(I\{X_{t,j} \leq q_j(\tau_2)\})_{t\in\mathbb{Z}}$.

Note that quantile coherency is a complex-valued, 2π-periodic function of the variable ω, and Hermitian in the sense that we have

$$\overline{\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)} = \mathfrak{R}^{j_1,j_2}(-\omega;\tau_1,\tau_2) = \mathfrak{R}^{j_2,j_1}(\omega;\tau_2,\tau_1) = \mathfrak{R}^{j_2,j_1}(2\pi+\omega;\tau_2,\tau_1).$$

Following similar arguments as in Section 2.1 of Birr et al. (2018), it can be shown that $\Re\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ describes the dynamics of the process switching between the j_1st component being below the τ_1-quantile and the j_2nd component being above the τ_2-

quantile. Consequently, for τ_1 close to 0 and for τ_2 close to 1 it describes the dynamics of changing from an extreme in one component to an extreme in another component. Further, it can be shown that $\Im\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ contains information about asymmetry.

A discussion of related quantities, how to interpret, how not to interpret them and how they are related to their traditional counterparts in the Gaussian case can be found in Sections S1, S2, and S3 of the supplementary material.

Recently, important contributions that aim at accounting for more general dynamics emerged in the literature. Measures as, for example, distance correlation Székely et al. (2007) and martingale difference correlation Shao and Zhang (2014) go beyond traditional correlation and instead can indicate whether random quantities are independent or martingale differences, respectively. For time series, in the time domain, Zhou (2012) introduced auto distance correlations that are zero if and only if the measured time series components are independent. Linton and Whang (2007), and Davis et al. (2009) introduced the (univariate) concepts of quantilograms and extremograms, respectively. More recently, quantile correlation Schmitt et al. (2015), and quantile autocorrelation functions Li et al. (2015) together with cross-quantilograms Han et al. (2016) have been proposed as a fundamental tool for analysing dependence in quantiles of the distribution.

In the frequency domain, Hong (1999) introduced a generalised spectral density. In the generalised spectral density covariances are replaced by quantities that are closely related to empirical characteristic functions. In Hong (2000) the Fourier transform of empirical copulas at different lags is considered for testing the hypothesis of pairwise independence. Recently, under the names of Laplace-, quantile and copula spectral density and spectral density kernels, various quantile-related spectral concepts have been proposed, for the frequency domain. The approaches by Hagemann (2013) and Li (2008, 2012) are designed to consider cyclical dependence in the distribution at user-specified quantiles. Mikosch and Zhao (2014, 2015) define and analyse a periodogram (and its integrated version) of extreme events. As noted by Hagemann (2013) other approaches aim at discovering "the presence of any type of dependence structure in time series data", referring to work of Dette et al. (2015) and Lee and Rao (2012). This comment also applies to Kley et al. (2016). In the present paper our aim is to generalise the existing approaches to multivariate time series. The extensions to the terminology that we provide, in particular the introduction of the standardised quantile coherency, is very important for economic applications, because it enables the analyst to perform a more detailed joint analysis of the serial and cross sectional dependence in multiple time series.

For the univariate case different approaches to consistent estimation were considered. Li (2008) proposed an estimator for a weighted version of the quantile spectra, based on least absolute deviation regression, for the special case where $\tau_1 = \tau_2 = 0.5$. Li (2012) generalised the estimator, using quantile regression, to the case where $\tau_1 = \tau_2 \in (0,1)$. The general case, in which the quantities can be related to the copulas of pairs, was first considered by Dette et al. (2015). These authors also were the first to consider a rank-based version of the quantile regression-type estimator which eliminates the need to estimates the weights in Li (2008, 2012). For the case where $\tau_1 = \tau_2 \in (0,1)$, Hagemann (2013) proposed a version of the traditional L^2-periodogram where the observations are replaced with $I\{\hat{F}_{n,j}(X_{t,j}) \le \tau\} = I\{R_{n;t,j} \le n\tau\}$. Kley et al. (2016) generalised this estimator, in the spirit of Dette et al. (2015), by considering cross-periodograms for arbitrary couples $(\tau_1,\tau_2) \in [0,1]^2$, and proved that it converges, as a stochastic process, to a complex-valued Gaussian limit. An estimator defined in analogy to the traditional lag-window estimator was analysed by Birr et al. (2017) in the context of non-stationary time series.

4. ASYMPTOTIC PROPERTIES OF THE PROPOSED ESTIMATORS

To derive the asymptotic properties of the estimators defined in Section 3 some assumptions need to be made. Recall (cf. Brillinger (1975), p. 19) that the rth order joint cumulant $\mathrm{cum}(Z_1, \ldots, Z_r)$ of the random vector $(Z_1, \ldots, Z_r)$ is defined as

$$\mathrm{cum}(Z_1, \ldots, Z_r) := \sum_{\{\nu_1,\ldots,\nu_p\}} (-1)^{p-1}(p-1)! E\Big[\prod_{j\in\nu_1} Z_j\Big] \cdots E\Big[\prod_{j\in\nu_p} Z_j\Big],$$

with summation extending over all partitions $\{\nu_1, \ldots, \nu_p\}$, $p = 1, \ldots, r$, of $\{1, \ldots, r\}$. Regarding the range of dependence of $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ we make the following assumption,

ASSUMPTION 4.1. *The process $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is strictly stationary and exponentially α-mixing, that is, there exists constants $K < \infty$ and $\rho \in (0, 1)$, such that*

$$\alpha(n) := \sup_{\substack{A\in\sigma(X_0,X_{-1},\ldots)\\ B\in\sigma(X_n,X_{n+1},\ldots)}} \big|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\big| \le K\rho^n, \quad n \in \mathbb{N}. \tag{4.11}$$

Further, to establish consistency of the estimates we consider sequences of weights that asymptotically concentrate around multiples of 2π,

ASSUMPTION 4.2. *The weights are defined as $W_n(u) := \sum_{j=-\infty}^{\infty} b_n^{-1} W(b_n^{-1}[u + 2\pi j])$, where $b_n > 0$, $n = 1, 2, \ldots$, is a sequence of scaling parameters satisfying $b_n \to 0$ and $nb_n \to \infty$, as $n \to \infty$. The weight function W is real-valued, even, has support $[-\pi, \pi]$, bounded variation, and satisfies $\int_{-\pi}^{\pi} W(u)du = 1$.*

Comments on the assumptions will follow in the end of this section. The main result of this section (Theorem 4.1) will legitimise $\hat{\boldsymbol{\mathfrak{R}}}_{n,R}(\omega; \tau_1, \tau_2)$ as an estimator of the quantile coherency $\boldsymbol{\mathfrak{R}}(\omega; \tau_1, \tau_2)$. Results that legitimise $\boldsymbol{I}_{n,R}(\omega; \tau_1, \tau_2)$ and $\hat{\boldsymbol{G}}_{n,R}(\omega; \tau_1, \tau_2)$ as estimators of the quantile cross-spectral density $\mathfrak{f}(\omega; \tau_1, \tau_2)$ are deferred to the supplementary material to not impair the flow of the paper. The legitimacy of the estimates follows from the fact that the estimators converge weakly in the sense of Hoffman-Jørgensen (cf. Chapter 1 of van der Vaart and Wellner (1996)). We denote this mode of convergence by $\Rightarrow$. The estimators under consideration take values in the space of (element-wise) bounded functions $[0, 1]^2 \to \mathbb{C}^{d\times d}$, which we denote by $\ell^\infty_{\mathbb{C}^{d\times d}}([0, 1]^2)$. While results in empirical process theory are typically stated for spaces of real-valued, bounded functions, these results transfer immediately by identifying $\ell^\infty_{\mathbb{C}^{d\times d}}([0, 1]^2)$ with the product space $\ell^\infty([0, 1]^2)^{2d^2}$. Note that the space $\ell^\infty_{\mathbb{C}^{d\times d}}([0, 1]^2)$ is constructed along the same lines as the space $\ell^\infty_{\mathbb{C}}([0, 1]^2)$ in Kley et al. (2016).
We are now ready to state the main result of this section.

THEOREM 4.1. *Let Assumptions 4.1 and 4.2 hold. Assume that the marginal distribution functions F_j, $j = 1, \ldots, d$ are continuous and that constants $\kappa > 0$ and $k \in \mathbb{N}$ exist, such that $b_n = o(n^{-1/(2k+1)})$ and $b_n n^{1-\kappa} \to \infty$. Assume that for some $\varepsilon \in (0, 1/2)$ we have*

$\inf_{\tau\in[\varepsilon,1-\varepsilon]} \mathfrak{f}^{j,j}(\omega;\tau,\tau) > 0$, *for all* $j = 1,\dots,d$. *Then, for any fixed* $\omega \in \mathbb{R}$,

$$\sqrt{nb_n}\Big(\hat{\mathfrak{R}}_{n,R}(\omega;\tau_1,\tau_2) - \mathfrak{R}(\omega;\tau_1,\tau_2) - \mathfrak{B}_n^{(k)}(\omega;\tau_1,\tau_2)\Big)_{(\tau_1,\tau_2)\in[\varepsilon,1-\varepsilon]^2} \Rightarrow \mathbb{L}(\omega;\cdot,\cdot), \quad (4.12)$$

in $\ell^\infty_{\mathbb{C}^{d\times d}}([\varepsilon,1-\varepsilon]^2)$, *where*

$$\Big\{\mathbb{L}(\omega;\tau_1,\tau_2)\Big\}_{j_1,j_2} := \frac{1}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}}\Big(\mathbb{H}_{1,2} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}\mathbb{H}_{1,1} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}\mathbb{H}_{2,2}\Big), \quad (4.13)$$

$$\Big\{\mathfrak{B}_n^{(k)}(\omega;\tau_1,\tau_2)\Big\}_{j_1,j_2} := \frac{1}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}}\Big(\boldsymbol{B}_{1,2} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}\boldsymbol{B}_{1,1} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}\boldsymbol{B}_{2,2}\Big) \quad (4.14)$$

and we have written $\mathfrak{f}_{a,b}$ *for the quantile cross-spectral density* $\mathfrak{f}^{j_a,j_b}(\omega;\tau_a,\tau_b)$ *as defined in* (2.2), $\boldsymbol{B}_{a,b} := \sum_{\ell=2}^k \frac{b_n^\ell}{\ell!}\int_{-\pi}^{\pi} v^\ell W(v)dv \frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell}\mathfrak{f}^{j_a,j_b}(\omega;\tau_a,\tau_b)$, *and* $\mathbb{H}_{a,b}$ *for* $\mathbb{H}^{j_a,j_b}(\omega;\tau_a,\tau_b)$; *a component of* $\mathbb{H}(\omega;\cdot,\cdot) := (\mathbb{H}^{j_1,j_2}(\omega;\cdot,\cdot))_{j_1,j_2=1,\dots,d}$ *defined as a centred,* $\mathbb{C}^{d\times d}$*-valued Gaussian process characterised by*

$$\begin{aligned}\mathrm{Cov}\,\big(\mathbb{H}^{j_1,j_2}(\omega;u_1,v_1), \mathbb{H}^{k_1,k_2}(\lambda;u_2,v_2)\big)& \\ = 2\pi\Big(\int_{-\pi}^{\pi} W^2(\alpha)\mathrm{d}\alpha\Big)\Big(&\mathfrak{f}^{j_1,k_1}(\omega;u_1,u_2)\mathfrak{f}^{j_2,k_2}(-\omega;v_1,v_2)\eta(\omega-\lambda) \\ &+ \mathfrak{f}^{j_1,k_2}(\omega;u_1,v_2)\mathfrak{f}^{j_2,k_1}(-\omega;v_1,u_2)\eta(\omega+\lambda)\Big), \quad (4.15)\end{aligned}$$

where $\eta(x) := I\{x = 0(\mod 2\pi)\}$ *[cf. (Brillinger, 1975, p. 148)] is the* 2π*-periodic extension of Kronecker's delta function. The family* $\{\mathbb{H}(\omega;\cdot,\cdot), \omega\in[0,\pi]\}$ *is a collection of independent processes and* $\mathbb{H}(\omega;\tau_1,\tau_2) = \overline{\mathbb{H}(-\omega;\tau_1,\tau_2)} = \mathbb{H}(\omega+2\pi;\tau_1,\tau_2)$.

The proof of Theorem 4.1 is lengthy and technical and therefore delegated to the online supplement (Section S6). Comparing Theorem 4.1 with results for the traditional coherency (see, for example, Theorem 7.6.2 in Brillinger (1975)) we observe that the distribution of $\hat{\mathfrak{R}}_{n,R}(\omega;\tau_1,\tau_2)$ is asymptotically equivalent to that of the traditional estimator [cf. (7.6.14) in Brillinger (1975)] computed from the unobserved time series

$$\big(I\{F_{j_1}(X_{t,j_1}) \le \tau_1\}, I\{F_{j_1}(X_{t,j_2}) \le \tau_2\}\big), \quad t = 0,\dots,n-1. \quad (4.16)$$

The convergence to a Gaussian process in (4.12) can be employed to obtain asymptotically valid pointwise confidence bands. To this end, the covariance kernel of $\mathbb{L}$ can easily be determined from (4.13) and (4.15), yielding an expression similar to (7.6.16) in Brillinger (1975). A more detailed account on how to conduct inference is given in Section S5 of the supplementary material. Note that the bound to the order of the bias given in (7.6.15) in Brillinger (1975) applies to the expansion given in (4.14).

If W is a kernel of order $p \ge 1$ we have that the bias is of order b_n^p. Thus, if we choose the mean square error minimising bandwidth $b_n \asymp n^{-1/(2p+1)}$ the bias will be of order $n^{-p/(2p+1)}$. Regarding the restriction $\varepsilon > 0$, note that the convergence (4.12) can not hold if (τ_1,τ_2) is on the border of the unit square, as the quantile coherency $\mathfrak{R}(\omega;\tau_1,\tau_2)$ is not defined if $\tau_j \in \{0,1\}$, as this implies that $\mathrm{Var}(I\{F_j(X_{t,j}) \le \tau_j\}) = 0$.

We now comment on the assumptions: Assumption 4.1 holds for a wide range of popular, linear and nonlinear processes. Examples (possibly, under mild additional assumptions) include the traditional VARMA or vector-ARCH models as well as many others. It is important to observe that Assumption 4.1 does not require the existence of any moments, which is in sharp contrast to the classical assumptions, where moments up to the order of the respective cumulants have to exist. Assumption 4.2 is quite standard in classical time series analysis [cf., for example, Brillinger (1975), p. 147].

5. QUANTILE CROSS-SPECTRAL ANALYSIS OF STOCK MARKET RETURNS: A ROUTE TO MORE ACCURATE RISK MEASURES?

Stock market returns belong to one of the prominent datasets in economics and finance. Although many important stylised facts about their behaviour have been established in the past decades, it remains a very active area of research. Despite the efforts, an important direction, which has not been fully addressed is stylised facts about the joint distribution of returns. Especially during the last turbulent decade, understanding the behaviour of joint quantiles in return distributions became particularly important, as it is essential for understanding systemic risk; "the risk that the intermediation capacity of the entire system can be impaired"; cf. Adrian and Brunnermeier (2016). Several authors focus on explaining tails of the bivariate market distributions in different ways. Adrian and Brunnermeier (2016) proposed to classify institutions according to the sensitivity of their quantiles to shocks to the market. Most closely related to the notion of how we view the dependence structures is the multivariate regression quantile model of White et al. (2015), which studies the degree of tail interdependence among different random variables directly.

Quantile cross-spectral analysis, as designed in this paper, allows to analyse the fundamental dependence quantities in the tails (but also in any other part) of the joint distribution and across frequencies. An application to stock market returns may therefore provide deeper insight about dependence in stock markets, and lead to a more powerful analysis securing us against financial collapses.

One of the important features of stock market returns is time variation in its volatility. Time-varying volatility processes can cross almost every quantile of their distribution (cf. Hagemann (2013)), and create peaks in quantile spectral densities as shown by Li (2014). These notions have recently been documented by Engle and Manganelli (2004) and Žikeš and Baruník (2016) who propose models for the conditional quantiles of the return distribution based on the past volatility. In the multivariate setting, strong common factors in volatility are found by Barigozzi et al. (2014) who conclude that common volatility is an important risk factor. Hence, common volatility should be viewed as a possible source of dependence. Because we aim to find the common structures in the joint distribution of returns, we study returns standardised by its volatility that we estimate by a GARCH(1,1) model; cf. Bollerslev (1986). This first step is commonly taken in the literature of modelling the joint market distribution using copulas; cf. Granger et al. (2006) and Patton (2012). In these approaches the volatility in the marginal distributions is modelled first, and the common factors are then considered in the second step. Consequently, this will allow us to discover other possible common factors in the joint distribution of market returns across frequencies, that result in spurious dependence, but which will not be overshadowed by the strong volatility process.

We choose to study the joint distribution of portfolio returns and excess returns on the broad market, hence looking at one of the most commonly studied factor structures in the literature as dictated by asset pricing theories; cf. Sharpe (1964) and Lintner (1965). As an excess return on the market, we use value-weighted returns of all firms listed on the NYSE, AMEX, or NASDAQ from the Center for Research in Security Price (CRSP)

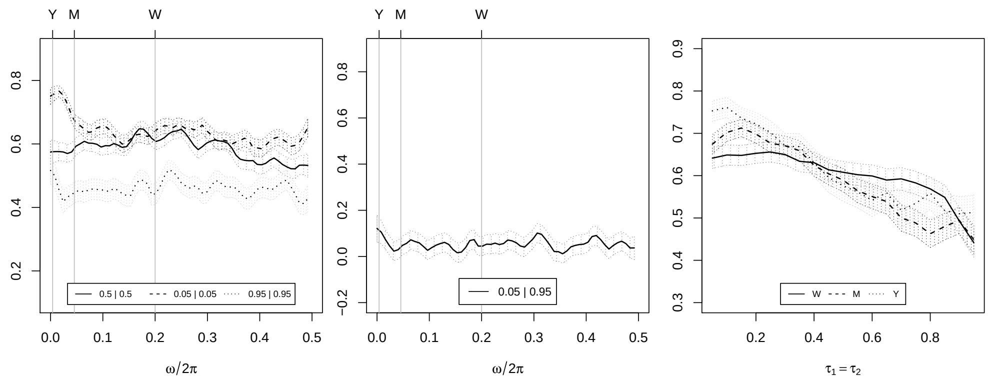

Figure 3. Quantile coherency estimates for the portfolio.

database. For the benchmark portfolio, we use an industry portfolio formed from consumer non-durables.[2] We used $n = 23385$ daily observations (from 1 July 1926 through to 30 June 2015). The data includes several crisis periods and therefore might not be suitable to be viewed as a strictly stationary time series. Nevertheless, we choose to study this long period of data as we believe that longer than yearly cycles might constitute an important possible source of dependence, and we believe the empirical results are practically interesting. Moreover, by standardising the returns by their volatility we removed what we believe is the most important source of time-variation in data.[3]

In the left panel of Figure 3, quantile coherency estimates for the $0.05|0.05$, $0.5|0.5$, and $0.95|0.95$ combinations of quantile levels of the joint distribution are shown for the industry portfolio and excess market returns over frequencies. The centre panel in Figure 3, on which we comment later, shows the $0.05|0.95$ combination. We have used the Epanechnikov kernel and a bandwidth of $b_n = 0.5n^{1/4}$ for the computation of the estimates (cf. (2.8)). The confidence intervals, shown as dotted regions, are at the 95% level and were constructed according to the procedure described in Section S5 of the supplementary material. For clarity, we plot the x-axis in daily cycles and also indicate the frequencies that correspond to yearly, monthly, and weekly periods. While we use daily data the highest possible frequency of 0.5 indicates 0.5 cycles per day (i. e., a 2-day period). While precise frequencies do not have an economic meaning, one needs to understand the interpretation with respect to the time domain. For example, a sampling frequency of 0.2 corresponds to 0.2 cycles per day translating to a 5 days period (equivalent to one week), but the frequency of 0.3 translates to a hardly interpretable $3.\overline{3}$ period. Hence, the upper label of the x-axis is of particular interest to an economist, as one can study how weekly, monthly, or yearly cycles are connected across quantiles of the joint distribution. For the clarity of presentation, we focus on the real part of the quantities, which relates

[2]Note to choice of the data: we use the publicly available data available and maintained by Fama and French at `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. This data set is popular among researchers, and while many types of portfolios can be chosen, we chose consumer non-durables randomly for this application. Although very interesting and attractive, it is far beyond the scope of this work to present and discuss results for wider portfolios formed on distinct criteria.

[3]As a robustness check, we have sliced the time series into decades and found that our results on non-overlapping windows do not materially change.

to the dynamics of the process switching between the j_1st component being below the τ_1-quantile and the j_2nd component being above the τ_2-quantile (cf. Section 2).

The real parts of the quantile coherency estimates reveal frequency dynamics in quantiles of the joint distribution of the returns under study. Generally, cycles at the lower quantiles appear to be more strongly dependent than at the upper quantiles, which is a well documented stylised fact about stock market returns. It points us to the fact that returns are more dependent during business cycle downturns, than upturn; cf. Erb et al. (1994), Longin and Solnik (2001), Ang and Chen (2002) and Patton (2012). More importantly, lower quantiles are more strongly related in periods longer than one week on average in comparison to shorter than weekly periods, and are even more connected at longer than monthly cycles. This suggests that infrequent clusters of large negative portfolio returns are better explained by excess market returns than small daily fluctuations. Returns in upper quantiles of the joint distribution seem to be connected similarly across all frequencies. The same result holds also for the median. For a better exposure, we also present quantile coherency estimates for three fixed weekly, monthly, and yearly periods (corresponding to $\omega \in 2\pi\{1/5, 1/22, 1/250\}$, respectively) at all quantile levels $\tau_1 = \tau_2 \in \{0.05, 0.1, \ldots, 0.95\}$ in the right panel of Figure 3. This alternative plot highlights the previous discussion.

We now compare our findings to a corresponding analysis with the cross-quantilogram, a related quantile-based measure for serial dependence in the time domain. Considering a strictly stationary, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$-valued time series $(y_{1t}, y_{2t}, x_{1t}, x_{2t})$, with $t \in \mathbb{Z}$ and $d_1, d_2 \in \mathbb{N}$, denoting the conditional distribution of the series y_{it} given x_{it} by $F_{y_i|x_i}(\cdot|x_{it})$, and the quantile function as $q_{i,t}(\tau_i) = \inf\{v : F_{y_i|x_i}(\cdot|x_{it}) \geq \tau_i\}$, $\tau_i \in (0, 1)$, $i = 1, 2$; Han et al. (2016) define the cross-quantilogram as

$$\rho_{(\tau_1,\tau_2)}(k) := \frac{\mathbb{E}\big[(I\{y_{1t} < q_{1,t}(\tau_1)\} - \tau_1)(I\{y_{2,t-k} < q_{2,t-k}(\tau_2)\} - \tau_2)\big]}{\Big(\mathbb{E}\big[(I\{y_{1t} < q_{1,t}(\tau_1)\} - \tau_1)^2\big]\mathbb{E}\big[(I\{y_{2,t-k} < q_{2,t-k}(\tau_2)\} - \tau_2)^2\big]\Big)^{1/2}}.$$

With no covariate information in our data example, this reduces to $x_{1t} = x_{2t} = 1$ and $q_{i,t}$ being the quantile of the marginal distribution of y_{it}. It is important to note that the cross-quantilogram is defined as a standardised measures of serial dependencies between the events $\{y_{1t} \leq q_{1,t}(\tau_1)\}$ and $\{y_{2t} \leq q_{2,t}(\tau_2)\}$ in the time domain, while quantile coherency is defined similarly, but in the frequency domain.

In Figure 4 we present the cross-quantilograms that we estimated from our data example. For the computation we have used the estimator and stationary bootstrap procedure defined in Han et al. (2016). More precisely, we used the implementation that is available in the R package **quantilogram**; cf. Han et al. (2014). Inspecting the plots, it can be seen that there are lags k, typically short, where significant dependence is present. Further, it is possible to guess that there is periodic variation of positive and negative dependence at the 0.05 quantile level, while at the 0.95 quantile level the dependence seems to be largely positive. Yet, taking into account the confidence intervals, it is uncertain if this is a significant pattern. Further, comparing the discussion of these periodic patterns shown by cross-quantilogram with what we were able to read from quantile coherency in Figure 3, it is difficult to read specific weekly, monthly and yearly periodic components and whether or not they are significant. Thus, at least in the specific case where a researcher is interested in the dependence of cycles, we believe that quantile coherency can provide a perspective that is unavailable in the time domain analysis.

To summarise the result of our empirical analysis: while asymmetry is commonly found

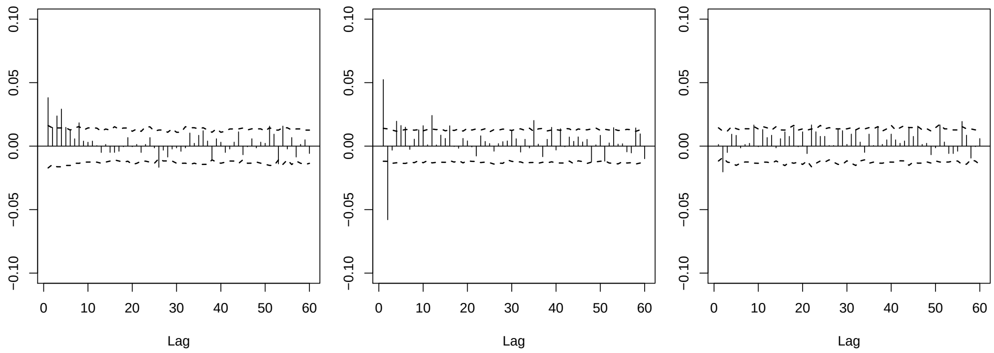

Figure 4. Cross-quantilogram estimates for the portfolio.

by researchers, we document frequency dependent asymmetry of stock market returns (i. e., asymmetry with respect to cycles in the joint distribution). In case this behaviour would be common across larger classes of assets, our results may have large implications for one of the cornerstones of asset pricing theory assuming normal distribution of returns. It leads us to the call for more general models, and more importantly to the need of restating the asset pricing theory in a way that allows to distinguish between short run and long run behaviour of investors.

Our results are also crucial for systemic risk measurement, as an investor wishing to optimise a portfolio should focus on stocks which will not be connected at lower quantiles, in a situation of distress, but will be connected at upper quantiles, in a situation of market upturns in a given investment period. We document behaviour which is not favourable to such an investor using traditional pricing theories, as we show that broad stock market returns contain a common factor more frequently during downturns than during upturns. This suggests that the portfolio at hand might be much riskier than it were implied by common measures. Further, our results suggest that this effect becomes even worse for long-run investors.

An important feature of our quantile cross-spectral measures is that they enable us to measure dependence also between $\tau_1 \neq \tau_2$ quantiles of the joint distribution. In the central panel of Figure 3 we document that the dependence between the $0.05|0.95$ quantiles of the return distribution is not very strong. Generally speaking, no intense dependence can be seen between large negative returns of the stock market, and large positive returns of the portfolio under study. This kind of analysis may be even more interesting in the case where dependence between individual assets is studied. There, negative news may have strong opposite impact on the assets under study.

Finally, some words of caution to the reader, about the interpretation of the quantities which we have estimated, are in order. In Section S3 of the supplementary material we provide a link between quantile coherency and traditional measures of dependence under the assumption of normally distributed data. The quantile-based measures are designed to capture general dependence types without restrictive assumptions on the underlying distribution of the process. Hence, here we have intentionally not relate it to traditional correlation which, ideally, should only be interpreted when the process is known to be Gaussian. The financial returns under study in this section are known to depart from normality. Therefore, quantile coherency is not directly comparable to traditional correlation measures. What we can see is generally strong dependence between the portfolio returns and excess market returns at all quantiles confirming the fact that excess returns are a strong common factor for the studied portfolio returns. The details that the quantile-based analysis in this section revealed would have remained hidden in an analysis based on the traditional coherency.

6. QUANTILE COHERENCY IN A MODEL ASSESSING EXERCISE

In the previous section we demonstrated how quantile coherency can be used by applied researchers to reveal cyclical features of the data that might remain invisible if the data is analysed solely with covariance-based dependency measures. In this section we illustrate how quantile coherency can be used to assess the capability of time series models to capture such cycles documented in the data.

More precisely, we fit several bivariate time series models and then compare the quantile coherencies implied by estimated parameters with those obtained from a non-parametric estimation (cf. Figure 3). The graphical approach of assessing the models is similar to the one proposed in Birr et al. (2018). For the sake of clarity, we focus on two classes of models: (a) vector autoregressive (VAR) models, and (b) vector versions of the quantile autoregressive (QVAR) model introduced by Koenker and Xiao (2006). Classical VAR used by many applied researchers assumes the same autoregressive structure at all quantiles. To model asymmetry, one can employ more flexible copulas allowing for asymmetric dependence. In addition, QVAR allows different autoregressive structure at different quantiles. Hence different quantiles can be driven by processes with different cyclical properties.

We discuss the models in order, from simple to more complex, and evaluate if the more complex models are better suited to capture the weekly, monthly and yearly cycles of quantile-related features which were discovered in the stock market returns analysis of Section 5.

We begin by fitting a VAR(1) to the stock market returns. The fitted model is

$$\begin{aligned} Y_{t,1} &= 0.0987 + 0.056 Y_{t-1,1} + 0.186 Y_{t-1,2} + \varepsilon_{t,1}, \\ Y_{t,2} &= 0.0369 - 0.056 Y_{t-1,1} + 0.175 Y_{t-1,2} + \varepsilon_{t,2}, \end{aligned} \tag{6.17}$$

where $(\varepsilon_{t,1}, \varepsilon_{t,2})$ is white noise with an estimated $\mathrm{Corr}(\varepsilon_{t,1}, \varepsilon_{t,2}) \approx 0.822$. Adding the common assumption that the $(\varepsilon_{t,1}, \varepsilon_{t,2})$ are independent and jointly Gaussian, the corresponding quantile coherencies can be determined. Quantile coherencies implied by the model (6.17) are depicted in the top row of Figure 5. For easier comparison, we consider the same combinations of frequencies and quantile levels as in Figure 3. In the picture it is clearly visible that dependencies of cycles implied by this Gaussian models are symmetric. For example, the dependence at the $0.05|0.05$ and at the $0.95|0.95$ level are equally strong for all frequencies. In contrast, the nonparametric estimate obtained from the data (cf. Figure 3) shows strong asymmetry. Further, we can see that for the weekly, monthly and yearly frequencies, which might be of particular interest for applied researchers, the dependencies at the $\tau|\tau$ and at the $1-\tau|1-\tau$ level coincide as well. If an applied researcher seeks to model dependencies as the ones revealed in Section 5, the Gaussian VAR model might therefore be too restrictive.

Next, we consider non-Gaussian versions of the fitted VAR. To obtain these models, note that the innovations in (6.17) are assumed to be white noise, but are not required

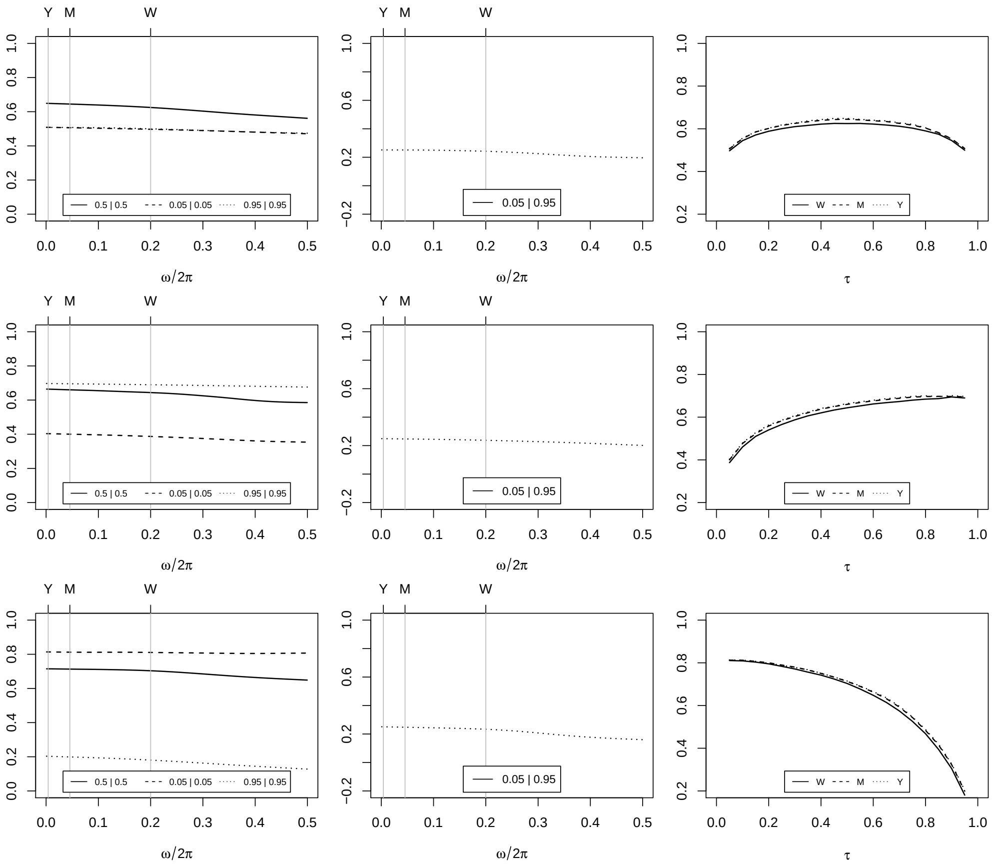

Figure 5. Quantile coherency simulated from the VAR models.

to be i. i. d. Gaussian. Another plausible model is therefore obtained by specifying any joint distribution for $(\varepsilon_{t,1}, \varepsilon_{t,2})$ that has first and second moment as implied by the fitted VAR model. For illustration we now consider the following two cases. In both cases we assume the marginal distributions to be standard normal. In the first case we assume that the dependence is according to a Clayton copula with parameter $\theta = 4$. In the second case we assume that it is according to a Gumbel copula with parameter $\theta = 2.7$. As one might expect, the dependence in the tails of the VAR(1) process is now remarkably different. As it can be seen from the middle-left plot in Figure 5, for the case of the Clayton copula there is stronger dependence in the lower tail (0.05|0.05) and weaker dependence in the upper tail (0.95|0.95). The dependence is slightly stronger for low frequencies, which is expected from the temporal dependence in the VAR model. In the bottom-left plot of Figure 5, on the other hand, we see stronger dependence in the upper and weaker dependence in the lower tail. Interestingly, as can be seen from the centre plots, the dependence of cycles in changing from being below the 0.05-quantile in the first component to being below the 0.95-quantile in the second component does not depend much on the choice of the copula. Finally, in the right plots of Figure 5, we see how the dependence changes according to the quantile level when cycles at the weekly, monthly and yearly frequencies, which we think might be most relevant to some practitioners, are considered. As expected, we see that for the case of the Clayton copula the dependence decreases as the quantile level τ increases, where for the case of the Gumbel copula the dependence increases if τ increases. Although the models with the Gumbel and Clayton copula capture asymmetric dependence better than the one with the Gaussian copula, we can still see that they depart from the data in terms of quantile coherency.

In the discussion before, we have seen three versions of a VAR(1) model, neither of which was particularly well suited to capture the type of dependence of cycles at quantile level which we observed in Section 5. In the second part of our modelling exercise we now turn our attention to a more flexible class of time series models. Motivated by the quantile autoregression model that was introduced by Koenker and Xiao (2006), we consider quantile vector autoregression, QVAR, a VAR model with random coefficients:

$$Y_{t,j} = \theta_{j0}(U_{t,j}) + \theta_{j1}(U_{t,j})Y_{t-1,1} + \theta_{j2}(U_{t,j})Y_{t-1,2}, \quad j = 1, 2, \tag{6.18}$$

where the θ_{ji} are coefficient functions and the $U_{t,j}$ are assumed to be independent and uniformly distributed on $[0, 1]$. Zhu et al. (2018) discuss a model similar to (6.18). Our aim here is to assess whether the time series model (6.18) is flexible enough to capture cyclical features in quantiles that were identified in Section 5. To this end, we choose the parameter functions in a data-driven way and then simulate the corresponding quantile coherency to compare with the the nonparametric estimate. Motivated by the estimation method in Zhu et al. (2018), we compute

$$\hat{\theta}(\tau) = \arg\min_{\theta(\tau)} \sum_{j=1}^{2} \sum_{t=2}^{n} \rho_\tau\big(Y_{t,j} - \theta_{j0}(\tau) - \theta_{j1}(\tau)Y_{t-1,1} - \theta_{j2}(\tau)Y_{t-1,2}\big), \tag{6.19}$$

$\tau \in \mathcal{T} := \{1/50, 2/50, \ldots, 48/50, 49/50\}$, where $\rho_\tau(u) := u(\tau - I\{u < \tau\})$ is the check function (cf. Koenker (2005)). For $\tau \notin \mathcal{T}$ we define $\hat{\theta}(\tau) := \hat{\theta}(\eta)$, $\eta := \arg\min_{\eta\in\mathcal{T}} |\tau - \eta|$ (choose the smaller η if there are two). The functions $\hat{\theta}(\tau) = (\hat{\theta}_{ji}(\tau))$, obtained from the stock market returns, are shown in Figure 7. It is interesting to observe that the functions $\hat{\theta}_{j1}$ and $\hat{\theta}_{j2}$, are not constant across quantile levels. This possibly indicates that a VAR model is too simple to capture the complicated dynamics present in the stock markets returns. The "shock" at time t to the jth equation is delivered by $\hat{\theta}_{j0}(U_{tj})$.

Koenker and Xiao (2006) and Zhu et al. (2018) establish conditions that ensure that quantile regressions, similar to (6.19), can be used to consistently estimate the parameter functions of the models in their papers. In particular, their model-defining equations (corresponding to (6.18) in our model) are assumed to be monotonically increasing in $U_{t,j}$. The monotonicity condition further implies a particularly convenient form for the conditional quantile function of $Y_{t,j}$ given $Y_{t-1,1}, Y_{t-1,2}$. Fan and Fan (2006) argue that the quantile regression estimate considered by Koenker and Xiao (2006) will be a consistent estimate for the argument of the minimum of a population version of the loss function, under some mild conditions. For $\hat{\theta}(\tau)$, defined in (6.19), this corresponds to being a consistent estimator for

$$\theta^*(\tau) = \arg\min_{\theta(\tau)} \sum_{j=1}^{2} \mathbb{E}\rho_\tau\big(Y_{t,j} - \theta_{j0}(\tau) - \theta_{j1}(\tau)Y_{t-1,1} - \theta_{j2}(\tau)Y_{t-1,2}\big).$$

Fan and Fan (2006) point out that additional conditions, such as the monotonicity condition, are necessary for $\theta^*(\tau)$ and $\theta(\tau)$ to coincide. These important arguments have to be taken into account when interpreting $\hat{\theta}(\tau)$ as an estimator for $\theta(\tau)$. Of course, data

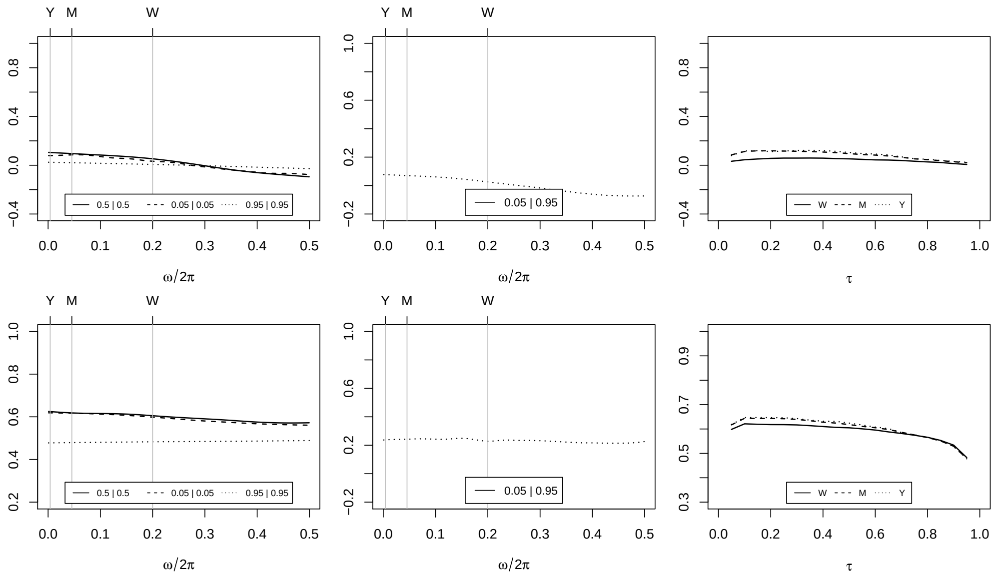

Figure 6. Quantile coherency simulated from several QVAR models.

can always be generated according to equation (6.18) where we substitute $\hat{\theta}(\tau)$ for $\theta(\tau)$. To assess whether the class of QVAR models is rich enough to reflect cyclical features in the quantiles as we have seen in the data in Section 5 it is sufficient to consider individual models from the class. For the purpose of this section, we select a QVAR model of the kind defined in (6.18), in a data-driven way, to then compare the implied quantile coherency with the one estimated non-parametrically in Section 5.

In the top row of Figure 6 the quantile coherencies associated with model (6.18) where $\hat{\theta}(\tau)$ was substituted for $\theta(\tau)$ are shown. The plots are of the same format as the ones we had considered before. Strikingly, we observe that the quantile coherency of the fitted model is substantially lower than what we see via the nonparametric estimate in Figure 3. Besides this, in the top row of Figure 6, we see that the general shape, decreasing lines with frequency, and ordering (0.95|0.95 shows less dependence than 0.05|0.05) resembles the nonparametric estimate more closely.

Finally, we propose to extend the QVAR(1) stated in (6.18), by adding spatial dependence. More precisely, the model we now consider is

$$\begin{aligned} Y_{t,1} &= \theta_{10}(U_{t,1}) + \theta_{111}(U_{t,1})Y_{t-1,1} + \theta_{121}(U_{t,1})Y_{t-1,2}, \\ Y_{t,2} &= \theta_{20}(U_{t,2}) + \theta_{211}(U_{t,2})Y_{t-1,1} + \theta_{221}(U_{t,2})Y_{t-1,2} + \theta_{210}(U_{t,2})Y_{t,1}. \end{aligned} \tag{6.20}$$

For this model, we compute quantile regression estimates

$$\begin{aligned} \hat{\theta}(\tau) = \arg\min_{\theta(\tau)} \Big(&\sum_{t=2}^{n} \rho_\tau\big(Y_{t,1} - \theta_{10}(\tau) - \theta_{111}(\tau)Y_{t-1,1} - \theta_{121}(\tau)Y_{t-1,2}\big) \\ &+ \sum_{t=2}^{n} \rho_\tau\big(Y_{t,2} - \theta_{20}(\tau) - \theta_{210}(\tau)Y_{t,1} - \theta_{211}(\tau)Y_{t-1,1} - \theta_{221}(\tau)Y_{t-1,2}\big)\Big). \end{aligned}$$

The estimates obtained from the stock returns data, that also should be cautiously interpreted, are depicted in Figure 8. Note that, if we substitute $Y_{1,t}$ in the second

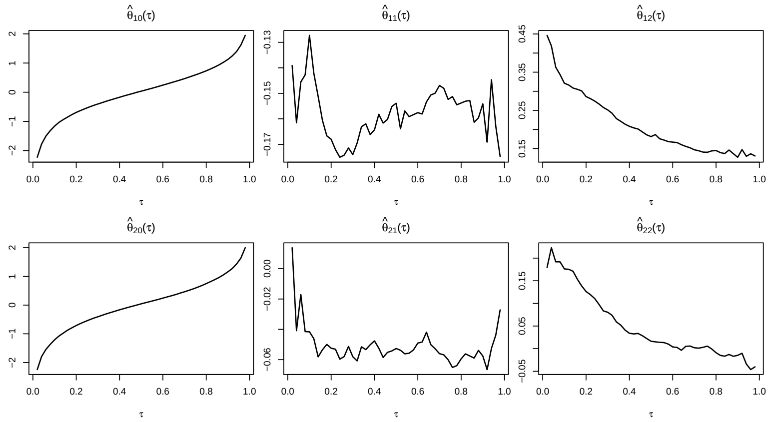

Figure 7. Estimated parameter functions for model (6.18).

equation of (6.20) by the expression given in the first equation, then we see that the "shocks" in this model are now dependent, as they are of the form $(\hat{\theta}_{10}(U_{t,1}), \hat{\theta}_{20}(U_{t,2}) + \hat{\theta}_{210}(U_{t,2})\hat{\theta}_{10}(U_{t,1}))$. The parameter function $\hat{\theta}_{210}$ moderates the strength of dependence. We now again look at the quantile coherency, depicted in the bottom row of Figure 6 and see that the quantile coherencies resemble the nonparameter estimates more closely (in shape, order and magnitude). This is true in particular for the right plot, where the frequency corresponding to the weekly, monthly, and yearly cycles are shown, which could be especially interesting for applied researchers.

In this section we illustrated how quantile coherency can be used by applied researchers to assess time series models regarding their capabilities to capture dependence between general cycles of stock market returns. We have seen that Gaussian VAR models are completely incapable of capturing asymmetries in the dependence of cycles. Our modelling exercise showed how non-Gaussian VAR models can possibly remedy this by allowing more general copulas for the errors in the model. Going further, we have also inspected bivariate quantile autoregression models and seen that their flexibility does better in capturing the general dependence between cycles that we have discovered using quantile coherency in Section 5.

7. CONCLUSION

In this paper we introduced quantile cross-spectral analysis of economic time series providing an entirely model-free, nonparametric theory for the estimation of general cross-dependence structures emerging from quantiles of the joint distribution in the frequency domain. We argue that complex dynamics in time series often arise naturally in many macroeconomic and financial time series, as infrequent periods of large negative values (lower quantiles of the joint distribution) may be more dependent than infrequent periods of large positive values (upper quantiles of the joint distribution). Moreover, the dependence may differ in the long-, medium, or short-run. Quantile cross-spectral analysis hence may fundamentally change the way how we view the dependence between economic time series, and may be viewed as a precursor to the subsequent developments in economic research underlying many new modelling strategies.

While connecting two branches of the literature which focus on the dependence bet-

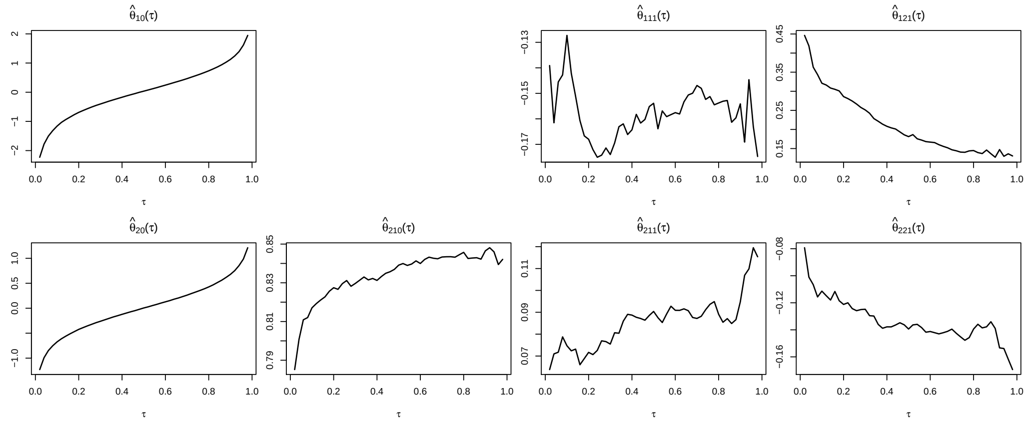

Figure 8. Parameter functions for model (6.20).

ween variables in quantiles of their joint distribution and across frequencies separately, the proposed methods may be viewed as an important step in robustifying the traditional cross-spectral analysis as well. Quantile-based spectral quantities are very attractive as they do not require the existence of moments, an important relaxation to the classical assumptions, where moments up to the order of the cumulants involved are typically assumed to exist. The proposed quantities are robust to many common violations of traditional assumptions found in data, including outliers, heavy tails, and changes in higher moments of the distribution. By considering quantiles instead of moments the proposed methods are able to reveal the dependence that remained invisible to the traditional toolsets. As an essential ingredient for a successful applications we have provided a rigorous analysis of the asymptotic properties of the introduced estimators and showed that for a general class of nonlinear processes, properly centred and smoothed versions of the quantile-based estimators converge to centred Gaussian processes.

In an empirical application, we have shown that classical asset pricing theories may not suit the data well, as commonly documented by researchers, because rich dependence structures exists varying across quantiles and frequencies in the joint distribution of returns. We document strong dependence of the bivariate returns series in periods of large negative returns, while positive returns display less dependence over all frequencies. This result is not favourable for an investor, as exactly the opposite would be desired: choosing to invest to stocks with independent negative returns, but dependent positive returns. Our tool reveals that systematic risk originates more strongly from lower quantiles of the joint distribution in the long-, and medium-run investment horizons in comparison to the upper quantiles. In a modelling exercise, we have illustrated how quantile coherency can be employed in the inspection of time series models and might help to find a model that is capable of capturing the dependencies of cycles of quantile-related features which we had previously revealed in our empirical application.

We believe that our work might open up many exciting new routes for future theoretical as well as empirical research. From the perspective of applications, exploratory analysis based on the quantile cross-spectral estimators can reveal new implications for improvement or even restating of many economic problems. Dependence in many economic time series is of a non-Gaussian nature, calling for an escape from covariance-based methods and allowing for a detailed analysis of the dependence in the quantiles of the joint distribution.

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REFERENCES

Adrian, T. and M. K. Brunnermeier (2016, July). CoVaR. *American Economic Review 106*(7), 1705–41.

Ang, A. and J. Chen (2002). Asymmetric correlations of equity portfolios. *Journal of Financial Economics 63*(3), 443–494.

Bae, K.-H., G. A. Karolyi, and R. M. Stulz (2003). A new approach to measuring financial contagion. *Review of Financial studies 16*(3), 717–763.

Barigozzi, M., C. Brownlees, G. M. Gallo, and D. Veredas (2014). Disentangling systematic and idiosyncratic dynamics in panels of volatility measures. *Journal of Econometrics 182*(2), 364–384.

Beaudry, P. and G. Koop (1993). Do recessions permanently change output? *Journal of Monetary economics 31*(2), 149–163.

Birr, S., T. Kley, and S. Volgushev (2018). Model assessment for time series dynamics using copula spectral densities: a graphical tool (arxiv:1804.01440). *ArXiv e-prints*.

Birr, S., S. Volgushev, T. Kley, H. Dette, and M. Hallin (2017). Quantile spectral analysis for locally stationary time series. *Journal of the Royal Statistical Society: Series B (Statistical Methodology) 79*(5), 1619–1643.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 31*(3), 307–327.

Brillinger, D. R. (1975). *Time Series: Data Analysis and Theory.* New York: Holt, Rinehart and Winston, Inc.

Croux, C., M. Forni, and L. Reichlin (2001). A measure of comovement for economic variables: Theory and empirics. *Review of Economics and Statistics 83*(2), 232–241.

Davis, R. A., T. Mikosch, et al. (2009). The extremogram: A correlogram for extreme events. *Bernoulli 15*(4), 977–1009.

Dette, H., M. Hallin, T. Kley, and S. Volgushev (2015). Of copulas, quantiles, ranks and spectra: An L_1-approach to spectral analysis. *Bernoulli 21*(2), 781–831.

Enders, W. and C. W. J. Granger (1998). Unit-root tests and asymmetric adjustment with an example using the term structure of interest rates. *Journal of Business & Economic Statistics 16*(3), 304–311.

Engle, R. F. and S. Manganelli (2004). CAViaR: Conditional autoregressive value at risk by regression quantiles. *Journal of Business & Economic Statistics 22*(4), 367–381.

Erb, C. B., C. R. Harvey, and T. E. Viskanta (1994). Forecasting international equity correlations. *Financial Analysts Journal 50*(6), 32–45.

Fan, J. and Y. Fan (2006). Comment. *Journal of the American Statistical Association 101*(475), 991–994.

Fan, Y. and A. J. Patton (2014). Copulas in econometrics. *Annual Review of Economics 6*(1), 179–200.

Granger, C. W. (2010). Some thoughts on the development of cointegration. *Journal of Econometrics 158*(1), 3–6.

Granger, C. W. and P. Newbold (1974). Spurious regressions in econometrics. *Journal of Econometrics 2*(2), 111–120.

Granger, C. W., T. Teräsvirta, and A. J. Patton (2006). Common factors in conditional distributions for bivariate time series. *Journal of Econometrics 132*(1), 43–57.

Granger, C. W. J. (1966). The typical spectral shape of an economic variable. *Econometrica 34*(1), 150–161.

Granger, C. W. J. (1969). Investigating causal relations by econometric models and cross-spectral methods. *Econometrica 37*(3), 424–438.

Hagemann, A. (2013). Robust spectral analysis (arxiv:1111.1965v2). *ArXiv e-prints*.

Han, H., O. Linton, T. Oka, and Y.-J. Whang (2014). *quantilogram: Quantilogramg*. R package version 0.1, retrieved from `https://sites.google.com/site/whangyjhomepage/Rcodes_CrossQuantilogram.zip` on 11 Dec. 2018.

Han, H., O. Linton, T. Oka, and Y.-J. Whang (2016). The cross-quantilogram: Measuring quantile dependence and testing directional predictability between time series. *Journal of Econometrics 193*(1), 251 – 270.

Hong, Y. (1999). Hypothesis testing in time series via the empirical characteristic function: a generalized spectral density approach. *Journal of the American Statistical Association 94*(448), 1201–1220.

Hong, Y. (2000). Generalized spectral tests for serial dependence. *Journal of the Royal Statistical Society Series B 62*(3), 557–574.

Kley, T. (2016). Quantile-based spectral analysis in an object-oriented framework and a reference implementation in R: The **quantspec** package (arxiv:1408.6755). *Journal of Statistical Software 70*(3), 1–27.

Kley, T., S. Volgushev, H. Dette, and M. Hallin (2016). Quantile spectral processes: Asymptotic analysis and inference. *Bernoulli 22*(3), 1770–1807.

Koenker, R. (2005). *Quantile Regression*. Econometric Society Monographs. Cambridge University Press.

Koenker, R. and Z. Xiao (2006). Quantile autoregression. *Journal of the American Statistical Association 101*(475), 980–990.

Lee, J. and S. S. Rao (2012). The quantile spectral density and comparison based tests for nonlinear time series (arxiv:1112.2759v2). *ArXiv e-prints*.

Li, G., Y. Li, and C.-L. Tsai (2015). Quantile correlations and quantile autoregressive modeling. *Journal of the American Statistical Association 110*(509), 246–261.

Li, T.-H. (2008). Laplace periodogram for time series analysis. *Journal of the American Statistical Association 103*(482), 757–768.

Li, T.-H. (2012). Quantile periodograms. *Journal of the American Statistical Association 107*(498), 765–776.

Li, T.-H. (2014). Quantile periodogram and time-dependent variance. *Journal of Time Series Analysis 35*(4), 322–340.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *The Review of Economics & Statistics*, 13–37.

Linton, O. and Y.-J. Whang (2007). The quantilogram: With an application to evaluating directional predictability. *Journal of Econometrics 141* (1), 250–282.

Longin, F. and B. Solnik (2001). Extreme correlation of international equity markets. *Journal of Finance*, 649–676.

Mikosch, T. and Y. Zhao (2014). A Fourier analysis of extreme events. *Bernoulli 20* (2), 803–845.

Mikosch, T. and Y. Zhao (2015). The integrated periodogram of a dependent extremal event sequence. *Stochastic Processes and their Applications 125* (8), 3126–3169.

Neftci, S. N. (1984). Are economic time series asymmetric over the business cycle? *The Journal of Political Economy*, 307–328.

Ning, C. Q. and L. Chollete (2009). The dependence structure of macroeconomic variables in the us. `http://www1.uis.no/ansatt/odegaard/uis_wps_econ_fin/uis_wps_2009_31_chollete_ning.pdf`. Technical report.

Patton, A. J. (2012). A review of copula models for economic time series. *Journal of Multivariate Analysis 110*, 4–18.

Schmitt, T. A., R. Schäfer, H. Dette, and T. Guhr (2015). Quantile correlations: Uncovering temporal dependencies in financial time series. *Journal of Theoretical and Applied Finance 18* (7), 16.

Shao, X. and J. Zhang (2014). Martingale difference correlation and its use in high-dimensional variable screening. *Journal of the American Statistical Association 109* (507), 1302–1318.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance 19* (3), 425–442.

Székely, G. J., M. L. Rizzo, and N. K. Bakirov (2007, 12). Measuring and testing dependence by correlation of distances. *Ann. Statist. 35* (6), 2769–2794.

van der Vaart, A. and J. Wellner (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. New York: Springer.

White, H., T.-H. Kim, and S. Manganelli (2015). VAR for VaR: Measuring tail dependence using multivariate regression quantiles. *Journal of Econometrics 187* (1), 169–188.

Xiao, Z. (2009). Quantile cointegrating regression. *Journal of Econometrics 150* (2), 248–260.

Zhou, Z. (2012). Measuring nonlinear dependence in time-series, a distance correlation approach. *Journal of Time Series Analysis 33* (3), 438–457.

Zhu, X., W. Wang, H. Wang, and W. K. Härdle (2018). Network quantile autoregression. *Available at SSRN: https://ssrn.com/abstract=3159671 or http://dx.doi.org/10.2139/ssrn.3159671.*

Žikeš, F. and J. Baruník (2016). Semi-parametric conditional quantile models for financial returns and realized volatility. *Journal of Financial Econometrics 14* (1), 185–226.